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# Finite Fractal Dimension of the Global Attractor for a Class of Non-Newtonian Fluids

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**Abstract**—We present a new criterion of finiteness of the fractal dimension of the attractor via the method of short trajectories developed in [1]. As an application, we deal with the so-called generalized Navier-Stokes equations characterized by nonlinear polynomial dependence of $(p-1)$ order between the stress tensor and the symmetric velocity gradient. We study the case $p \geq 2$ subject to space-periodic boundary conditions.

The existence of the global attractor with finite fractal dimension is then obtained in the following cases:

(i) in two dimensions if $p \geq 2$, and
(ii) in three dimensions if $p \geq 11/5$.



**Keywords**—Fractal dimension, Global attractor, Non-newtonian fluid, Generalized Navier-Stokes equations, Shear-dependent viscosity.

## 1. INTRODUCTION

For a given well-posed evolutionary problem in $X$, $X$ being a Banach space, $S_t : X \to X$ denotes its solution operator, which assigns to the initial value the (uniquely defined) value of the solution in the time $t$.

Recall that an unempty set $\mathcal{A} \subset X$ is called *global attractor* associated with the semigroup $\{S_t\}_{t\geq 0}$ if and only if

(i) $\mathcal{A}$ is compact,
(ii) $\mathcal{A}$ is invariant ($S_t\mathcal{A} = \mathcal{A}$ for all $t \geq 0$), and
(iii) $\mathcal{A}$ attracts bounded subsets of $X$, i.e., for any $B_0 \subset X$ bounded $\operatorname{dist}_X(S_t B_0, \mathcal{A}) \to 0$ as $t \to \infty$.

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By $d_f(\mathcal{A})$, we mean the fractal dimension of $\mathcal{A}$ defined as limes superior of $\log N_\varepsilon(\mathcal{A})$ versus $\log 1/\varepsilon$ as $\varepsilon \to 0$, where $N_\varepsilon(\mathcal{A})$ denotes the minimal number of $\varepsilon$-balls needed to cover $\mathcal{A}$. If $d_f(\mathcal{A})$ is finite, say $d_f(\mathcal{A}) < m/2$, $m \in \mathbb{N}$, then there exists an injective projection $F : \mathcal{A} \mapsto \mathbb{R}^m$ such that its inverse is Hölder continuous. This result that strengthens the results in [2] is due to Foias and Olson, cf. [3]. In another words, if $d_f(\mathcal{A})$ is finite, then the attractor $\mathcal{A}$ is located at the graph of a Hölder continuous mapping of a compact set in $\mathbb{R}^m$.

Let $\Omega = (0,1)^n$, $n = 2$ or 3, be an $n$-dimensional cube, $I = (0,T)$, $T > 0$. We study the following problem:

$$\begin{gathered} \operatorname{div} \boldsymbol{u} = 0, \\ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial \pi}{\partial x_i} + \frac{\partial \tau_{ij}^V}{\partial x_j} + f_i, \qquad i = 1, \dots, n. \end{gathered} \tag{1.1}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(x), \qquad \boldsymbol{x} \in \Omega, \quad \operatorname{div} \boldsymbol{u}_0 = 0. \tag{1.2}$$

$$\boldsymbol{u}, \pi \text{ are 1-periodic at each direction and have zero mean value.} \tag{1.3}$$

We use standard notation of function spaces: $H$ denotes the closure, with respect to the $L^2$-norm, of the space of smooth 1-periodic divergence-free functions with mean value zero, and $V_p$ denotes the closure of the same space with respect to the $W^{1,p}$-norm. For further notation, see [1, p. 500] or [4], if needed.

The class of investigated fluids is characterized by the polynomial dependence of the stress tensor $\tau^V$ on the symmetric velocity gradient $D(\boldsymbol{v})$. Its growth and coercivity are characterized by the parameter $p$. We study the case $p \geq 2$ and are mainly interested in three-dimensional flows.

The precise assumptions on the relation between $\tau^V$ and $D(\boldsymbol{v})$ can be found in [1] or [4], while here we put some rather typical examples:

(a) $\tau^V = 2\nu_0(1 + \varepsilon|D(\boldsymbol{v})|^{p-2})D(\boldsymbol{v})$, $p > 2$,
(b) $\tau^V = 2\nu_0 D(\boldsymbol{v})$,
(c) $\tau^V = 2\nu_0(1 + \xi(|D(\boldsymbol{v})|))D(\boldsymbol{v})$,

where in the case (c), we take for $\xi(\cdot)$ any nonnegative nondecreasing bounded function. If $p = 3$ and $\boldsymbol{v}$ denotes the large scale (average) velocity, then the model (a) coincides with the Smagorinsky model of turbulence (cf. [5]).

Note that in cases (b), (c), $p = 2$. The case (b) leads to the Navier-Stokes system, for which the well-posedness is an open problem. Thus, particularly in three dimensions, we can view model (a) as a natural perturbation of the Navier-Stokes system (cf. [6]). Despite the three-dimensional Navier-Stokes system, its modified version enjoys the existence of unique solution certainly for $p \geq (3n+2)/(n+2)$.

More precisely, if $\boldsymbol{u}_0 \in H$, then uniqueness holds (only) for $p \geq (n+2)/2$ (note that $(3n+2)/(n+2) = (n+2)/2$ if $n = 2$) and the existence of the attractor $\mathcal{A} \subset H$ is not a difficult task (see Section 2 for references or [7] for a detailed description). The uniqueness for $p \in [11/5, 5/2)$ is based on regularity results that require smoother initial data, e.g., $\boldsymbol{u}_0 \in V_p$. Construction of the attractor in $H$ in this range of $p$'s has been left open (Problem A).

Assume for a while that we have the attractor $\mathcal{A} \subset H$ for $p \geq (3n+2)/(n+2)$. The question is to show that $d_f(\mathcal{A})$ is finite. The two known methods developed earlier by Ladyzhenskaya (cf. [8]) and by Constantin and Foias (cf. [9]) to prove finiteness of the fractal dimension for the two-dimensional Navier-Stokes equations seem to be not applicable here because of the nonlinear elliptic operator (Problem B).

In order to overcome obstacles connected with Problems A and B, the concept of short trajectory was introduced in [1]. For Problem A, the set of all short trajectories is used as the new phase space (instead of $H$), and after defining the new dynamical system on this set in a natural way, the existence of a global finite-dimensional attractor with respect to this new dynamical system has been proved. To Problem B, i.e., to the case where the global attractor $\mathcal{A}$ exists, it has been also shown in [1] that the set of all $\delta$ trajectories, starting from $\mathcal{A}$, has finite fractal dimension with respect to the $L^2(0,\delta;H)$ norm.

The main observation of this paper is the following: having finite-dimensional attractor $\mathcal{A}^s_\delta \subset L^2(0,\delta;H)$, we consider the set consisting of end-points of short trajectories belonging to $\mathcal{A}^s_\delta$. Studying the properties of the so-obtained set, we obtain the following result.

THEOREM 1.1. *Let $p \geq (3n+2)/(n+2)$. Then there exists $\mathcal{A} \subset H$ such that*

(i) *$\mathcal{A}$ is compact in $H$ and bounded in $V_p$,*
(ii) *$S_t\mathcal{A} = \mathcal{A}$ for all $t \geq 0$,*
(iii) *for any bounded $B \subset H$ $\sup_{\gamma\in\Gamma_B} \operatorname{dist}_H(\chi_\gamma(t), \mathcal{A}) \to 0$ as $t \to \infty$, where $\{\chi_\gamma\}_{\gamma\in\Gamma_B}$ is the set of all trajectories with initial value in $B$, and*
(iv) *$d_f(\mathcal{A}) < \infty$.*

If $p \geq (n+2)/2$, then the property (iii) can be replaced by saying that $\mathcal{A}$ attracts all bounded sets in $H$. Proof of the theorem is sketched in Sections 3 and 4.

It reveals that this method provides a new way of constructing global attractors and estimating their fractal dimension. The method seems to be widely applicable to various dissipative systems. The computation of explicit bounds on fractal dimension of $\mathcal{A}$ is in progress. An interesting question is to compare the explicit bounds for the two-dimensional Navier-Stokes equations obtained by this new method and by the method of Lyapunov exponents.

## 2. A SURVEY OF KNOWN GLOBAL RESULTS

The aim of this section is to collect earlier proved results on global existence, uniqueness, regularity, and large time behavior regarding the system (1.1)–(1.3).

THEOREM 2.1. *Let $\boldsymbol{u}_0$, $\boldsymbol{f} \in H$. If $p \geq (3n+2)/(n+2)$, then there exists weak solution $\boldsymbol{u}$ to (1.1)–(1.3) such that*
$$\boldsymbol{u} \in C([0,T];H) \cap L^p(I;V_p)\,.$$
*If $p \geq (n+2)/2$, the solution is unique.*

PROOF. See [6], and the references therein; see also [10]. ∎

THEOREM 2.2. (HIGHER REGULARITY AND SMOOTHING PROPERTY).

(a) *If $n=2$ and $p \geq 2$, then $\boldsymbol{u} \in L^\infty_{\mathrm{loc}}(0,T;V_p)$,*
(b) *if $n=3$ and $p \geq 11/5$, then $\boldsymbol{u} \in L^\infty_{\mathrm{loc}}(0,T;V_p) \cap L^p_{\mathrm{loc}}(0,T;W^{1,3p}(\Omega)^3)$.*

*If $\boldsymbol{u}_0 \in V_p$, these results hold globally and the solution is unique also for $p \in [11/5, 5/2)$ if $n=3$.*

PROOF. See [6] for (a), [4,11,12] for (b). ∎

THEOREM 2.3. *If $p \geq (n+2)/2$, then the solution operators $\{S_t\}_{t\geq 0}$ possess the global attractor.*

PROOF. See [13] for $n=2$, [1] or [13] for $n=3$. ∎

Finally, let us, for the reader's convenience, sum up the results about continuity of solution operators $S_t$, which follow from the uniqueness proofs.

THEOREM 2.4. *If $p \geq (n+2)/2$, then the solution operators $S_t : H \mapsto H$ are locally Lipschitz continuous.*

*In the case $n=3$, $p \in [11/5, 5/2)$, the following can be said: let $\boldsymbol{u}$, $\boldsymbol{v}$ be two solutions, starting at $\boldsymbol{u}_0$, $\boldsymbol{v}_0 \in H$ respectively, let moreover $\boldsymbol{v}_0 \in W^{1,p}(\Omega)^3$. Then for $t \in [0,T]$*

$$\|\boldsymbol{u}(t) - \boldsymbol{v}(t)\|_H \leq C_1 \|\boldsymbol{u}_0 - \boldsymbol{v}_0\|_H, \qquad C_1 = C_1\left(\|\boldsymbol{v}_0\|_{1,p}\right). \tag{2.1}$$

PROOF. See [6] for $p \geq (n+2)/2$ and [1] for $n=3$, $p \in [11/5, 5/2)$. ∎

REMARK. Throughout the whole paper, we deal with the space-periodic problem, but this restriction is only because of the statement in Theorem 2.2 (b), which at the present time is known to be valid for $p$ slightly worse than 11/5. Without any doubts, if $p \geq 2$ for $n = 2$ and if $p \geq 12/5$ for $n = 3$, Theorem 1.1 holds for the Dirichlet boundary conditions, too.

## 3. CASE $p \geq (n+2)/2$, $n = 2$ OR 3

In this case, the global attractor to our semigroup $\mathcal{A}$ exists by Theorem 2.3. Let us denote by $\mathcal{A}_\delta$ the set of all $\delta$-trajectories starting from $\mathcal{A}$, i.e.,

$$\mathcal{A}_\delta = \{\chi : [0,\delta] \mapsto H;\ \chi(0) \in \mathcal{A};\ \chi \text{ is a weak solution}\}. \tag{3.1}$$

THEOREM 3.1. *Let $\delta$ be small enough and $p \geq (n+2)/2$. Then the set $\mathcal{A}_\delta$ has finite fractal dimension with respect to the norm $L^2(0,\delta;H)$.*

PROOF. We present here only the proof of the key inequality which reads: there exist $\delta > 0$ and $\kappa > 0$ such that for all $t \geq \delta$ and all $\chi^1$, $\chi^2 \in \mathcal{A}_\delta$ it holds

$$\int_0^\delta \left\|\nabla\chi^1(t+s) - \nabla\chi^2(t+s)\right\|_2^2 ds \leq \kappa \int_0^\delta \left\|\chi^1(s) - \chi^2(s)\right\|_2^2 ds. \tag{3.2}$$

Let $\chi^1$, $\chi^2$ be two $\delta$-trajectories from $\mathcal{A}_\delta$. Then by standard manipulation, we see that $\chi^1 - \chi^2$ satisfies

$$\frac{d}{dt}\left\|\chi^1 - \chi^2\right\|_2^2 + c_0\left\|\nabla\left(\chi^1 - \chi^2\right)\right\|_2^2 \leq c_1\|\nabla v\|_p^{2p/(2p-3)}\left\|\chi^1 - \chi^2\right\|_2^2. \tag{3.3}$$

Let $s \in (0,\delta)$, $r \in (s, 2\delta)$, and $t \geq \delta$. Set $\gamma(t) \equiv \sup_{\tau\in(s,2\delta)}\|(\chi^1 - \chi^2)(t+\tau)\|_2^2$. Then we integrate (3.3) between $t+s$ and $t+r$ and estimate the right-hand side by $\gamma(t)$. Also observe that due to Theorem 2.2, $\mathcal{A}$ is subset of bounded set in $V_p$, i.e., $\|\nabla\chi(s)\|_p^{2p/(2p-3)} \leq \rho$ for all $s \in [0,\delta]$ and for all $\chi \in \mathcal{A}_\delta$. This yields

$$\begin{aligned}\left\|\left(\chi^1 - \chi^2\right)(t+r)\right\|_2^2 &+ c_0\int_s^r \left\|\nabla\left(\chi^1 - \nabla\chi^2\right)(t+\tau)\right\|_2^2 ds \\ &\leq 2c_1\rho\delta\gamma(t) + \left\|\left(\chi^1 - \chi^2\right)(t+s)\right\|_2^2.\end{aligned} \tag{3.4}$$

Therefore,

$$\gamma(t) + c_0\int_\delta^{2\delta} \left\|\nabla\chi^1(t+\tau) - \nabla\chi^2(t+\tau)\right\|_2^2 ds \leq 4c_1\rho\delta\gamma(t) + 2\left\|\left(\chi^1 - \chi^2\right)(t+s)\right\|_2^2. \tag{3.5}$$

Now, we see that (3.2) follows from (3.5) by choosing $\delta < 1/(4c_1\rho)$ and integrating (3.5) with respect to $s$ between 0 and $\delta$.

The further steps of the proof are explained in [1] in detail. ∎

Since trajectories $\chi$ in $\mathcal{A}_\delta$ are $H$-continuous, we can define mapping $K : \mathcal{A}_\delta \mapsto H$ by

$$K(\chi) := \chi(\delta), \qquad \chi \in \mathcal{A}_\delta. \tag{3.6}$$

Then the mapping $K$ has the following elementary, but key properties.

PROPOSITION 3.2.

(i) *$K(\mathcal{A}_\delta) = \mathcal{A}$,*
(ii) *$K$ is Lipschitz continuous.*

PROOF. First, (i) is a consequence of invariance $\mathcal{A}$. To prove (ii), let us choose $\chi_1$, $\chi_2 \in \mathcal{A}_\delta$, and $t \in (0,\delta)$. We can write

$$\begin{aligned} \|K(\chi_1) - K(\chi_2)\|_H^2 &= \|\chi_1(\delta) - \chi_2(\delta)\|_H^2 = \|S_{\delta-t}\chi_1(t) - S_{\delta-t}\chi_2(t)\|_H^2 \\ &\overset{(2.5)}{\leq} C_1\|\chi_1(t) - \chi_2(t)\|_H^2. \end{aligned} \tag{3.7}$$

Lipschitz continuity of $K$ is then obtained by integrating (3.7) with respect to $t$ between 0 and $\delta$. ∎

Since $d_f(\mathcal{A}_\delta) < \infty$ and the fractal dimension does not grow under Lipschitz continuous mapping, the proof of Theorem 1.1 is complete in the case when $p \geq (n+2)/2$. We can summarize the result in the following statement.

COROLLARY 3.3. *The global attractor $\mathcal{A}$ has finite fractal dimension.*

## 4. CASE $p \in [11/5, 5/2)$, $n = 3$

In this case, we have existence of weak solutions by Theorem 2.1, but uniqueness is granted only for more regular data, $u_0 \in H \cap V_p$.

The proof of Theorem 1.1 is again approached via short trajectories. Denote by $\mathcal{X}_\delta$ the set of all $\delta$-trajectories, starting from $H$, i.e.,

$$\mathcal{X}_\delta := \{\chi : [0,\delta] \mapsto H;\ \chi \text{ is weak solution}\}. \tag{4.1}$$

Since we have no uniqueness, more trajectories can start from one point $u_0 \in H$. On the other hand, according to Theorem 2.2, any trajectory, once it has started, immediately becomes more regular (namely, it is bounded in $L^\infty_{\mathrm{loc}}(0,\delta;V_p)$) and consequently, the operators $L_t : \mathcal{X}_\delta \mapsto \mathcal{X}_\delta$ are well defined:

$$(L_t\chi)(s) := S_t\chi(s), \qquad s \in (0,\delta). \tag{4.2}$$

The operators $\{L_t\}_{t\geq 0}$ have semigroup property and are locally Lipschitz continuous with respect to the norm $L^2(0,\delta;H)$ (see [1], Lemma 4.4). In order to be sure that our space is complete, we consider the closure of the set $\mathcal{X}_\delta$ with respect to the norm $L^2(0,\delta;H)$, which is denoted by $\mathcal{X}_\delta^s$. Due to the continuity, $L_t$ can be naturally extended to the set $\mathcal{X}_\delta^s$. This set, equipped with the norm $L^2(0,\delta;H)$, serves now as a new phase space to the dynamical system, described by system (1.1). The following theorem holds.

THEOREM 4.1. *The semigroup $\{L_t\}_{t\geq 0}$ possesses the global attractor $\mathcal{A}_\delta^s \subset \mathcal{X}_\delta^s$. Moreover, if $\delta$ is small enough, then $d_f(\mathcal{A}_\delta^s) < \infty$.*

PROOF. The proof is again based on the inequality similar to (3.2). The details are in [1, Theorem 4.14]. ∎

Now we define operators $K : \mathcal{A}_\delta^s \mapsto H$ by

$$K(\chi) := \chi(\delta). \qquad \chi \in \mathcal{A}_\delta \tag{4.3}$$

and show that the set $K(\mathcal{A}_\delta^s)$ has the properties (i)–(iv) of Theorem 1.1. To do this, we first show better regularity of the set $\mathcal{A}_\delta^s$ and then we prove the proposition analogous to Part (ii) of the Proposition 3.2.

PROPOSITION 4.2. *The set $\mathcal{A}_\delta^s$ is bounded in $L^\infty(0,\delta;V_p)$ and $\mathcal{A}_\delta^s \subset \mathcal{X}_\delta$.*

PROOF. Let $\chi \in \mathcal{A}_\delta^s$. Since $L_\delta\mathcal{A}_\delta^s = \mathcal{A}_\delta^s$, there exists $\chi_0 \in \mathcal{A}_\delta^s$ such that $L_\delta\chi_0 = \chi$. By the definition, there exists a sequence $\chi_n \subset \mathcal{X}_\delta$, $\chi_n \to \chi_0$, such that $L_\delta\chi_n \to \chi$.

Without loss of generality, $\chi_n$ is bounded by some constant $C$, not depending on $\chi$, $\chi_0$. By contradiction, for every $n$ there exists $s_n \in (0,\delta/2)$ such that $\|\chi_n(s_n)\|_H \leq \sqrt{2/\delta}C$. Theorem 2.2 (b) now implies that the sequence $L_\delta\chi_n$ is bounded in $L^\infty(0,\delta;V_p)$.

Consequently, $\chi \in L^\infty(0,\delta;V_p)$. This estimate also allows us to pass the limit in the equation to obtain $\chi \in \mathcal{X}_\delta$. The proof of the proposition is finished. ∎

PROPOSITION 4.3. *There exists a constant $C > 0$ such that for any $\chi \in \mathcal{X}_\delta$, $\chi_0 \in \mathcal{A}^s_\delta$,*

$$\|K(\chi) - K(\chi_0)\|_H \leq C\|\chi - \chi_0\|_{L^2(0,\delta;H)}.$$

PROOF. Note that $K$ is well defined on $\mathcal{X}_\delta$ and consequently, due to the Proposition 4.2, also on $\mathcal{A}^s_\delta$. Now, let $t \in (0, \delta)$. Using (2.1) and the regularity of $\mathcal{A}^s_\delta$,

$$\|K(\chi) - K(\chi_0)\|^2_H = \|\chi(\delta) - \chi_0(\delta)\|^2_H \leq C_1\|\chi(t) - \chi_0(t)\|^2_H.$$

Then we continue as in the proof of Proposition 3.2. ∎

Now we can prove Theorem 1.1. Let us define $\mathcal{A} := K(\mathcal{A}^s_\delta)$. Since $K$ is Lipschitz continuous and $\mathcal{X}^s_\delta$ is compact and has finite fractal dimension, $\mathcal{A}$ has the same properties. Since $\mathcal{A}^s_\delta$ is bounded in $L^\infty(0, \delta; V_p)$, $\mathcal{A}$ is bounded in $V_p$. The invariance of $\mathcal{A}$ with respect to $S_t$ is a consequence of invariance $\mathcal{A}^s_\delta$ with respect to $L_t$.

It remains to show the attracting property (iii). Let $B \subset H$ be bounded set, let $B_\delta$ be the set of all $\delta$-trajectories, starting from $B$. Due to Proposition 4.3 we have

$$\sup_{\gamma\in\Gamma_B} \operatorname{dist}_H(\chi_\gamma(t+\delta), \mathcal{A}) = \operatorname{dist}_H(K(L_tB_\delta), K(\mathcal{A}^s_\delta)) \leq C \operatorname{dist}_{\mathcal{X}^s_\delta}(L_tB_\delta, \mathcal{A}^s_\delta).$$

Since $B_\delta$ is bounded (by *a priori* estimates) in $\mathcal{X}^s_\delta$, this tends to zero as $t$ goes to infinity. The proof of the theorem is finished. ∎

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